Phase Space

Theoretical Physics I, Mechanics · lecture notes

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Overview. These notes introduce *phase space*, the arena in which the whole of mechanics, and later statistical and quantum physics, is most naturally formulated. The single idea is that a mechanical system is described not by its position alone but by a point in a space of *states*; the laws of motion turn that space into a flow; and because energy is conserved, every possible motion of a one-dimensional system can be read off as a level curve, even when the equation of motion cannot be solved. We build directly on Newton's law, conservative forces, and the harmonic oscillator.

1 The state, the phase space, and the flow

The previous lesson introduced conservative forces and energy conservation through two examples we carry over here: the pendulum, with gravitational (height) potential $U(\theta) = mgL(1 - \cos\theta)$, and the harmonic oscillator, $U(q) = \frac{1}{2}m\omega^2 q^2$. A particle of mass m on a line obeys Newton's second law $m\ddot{q} = F(q)$, and in one dimension every position-dependent force is conservative, $F = -U'(q)$, with conserved energy $E = \frac{1}{2}m\dot{q}^2 + U(q) = p^2/2m + U(q)$, where $p \equiv m\dot{q}$. Newton's law is second order, so to predict the motion we must give, at one instant, a position *and* a velocity. Pairing q with the momentum $p = m\dot{q}$ puts position and velocity on the same footing, and the second-order law splits into a pair of first-order ones,

$$\boxed{\dot{q} = \frac{p}{m}, \qquad \dot{p} = F(q) = -U'(q).}$$

Nothing has been solved: we have merely traded one second-order equation for two first-order ones, but the gain is conceptual.

Definition 1 (State and phase space)**.** The pair $x = (q, p)$ is the *state* of the particle, and the set of all states $\Gamma = \{(q, p)\}$ is its *phase space*, here a plane. A single point of Γ is a complete "identity card": where the particle is *and* where it is heading. (For f degrees of freedom $\dim\Gamma = 2f$, always even.)

We keep the momentum $p = m\dot{q}$ rather than the velocity because it is conserved in collisions and is the natural partner of q in every later formulation; here the two differ only by the factor m. Already in one dimension Γ need not be a flat plane: for the oscillator the position is unrestricted, so $\Gamma = \mathbb{R}^2$; for the pendulum the angle θ and $\theta + 2\pi$ are the same configuration, so the line of positions closes into a circle and $\Gamma = S^1 \times \mathbb{R}$, a cylinder.

A useful freedom. The coordinate q need not be a Cartesian position: any variable that fixes the configuration will do (an angle, a separation, a normal-mode amplitude), with p the momentum *conjugate* to that choice. The pendulum below is the first example, its natural coordinate being the angle θ. Choosing such coordinates well is the point of the *Lagrangian* formulation later in the course.

Finally, write the equations of motion compactly as $\dot{x} = \mathbf{V}(x)$ with $\mathbf{V}(x) = (p/m,\ F(q))$: at every point the dynamics attaches a vector, a *vector field*, the phase velocity, like a stationary wind across Γ. A solution is a curve everywhere tangent to it (a phase curve, orbit, or trajectory), the path of a speck carried by the wind (Figure 1). We read this off in two examples; what makes the field so powerful, the existence–uniqueness theorem, we keep for the end.

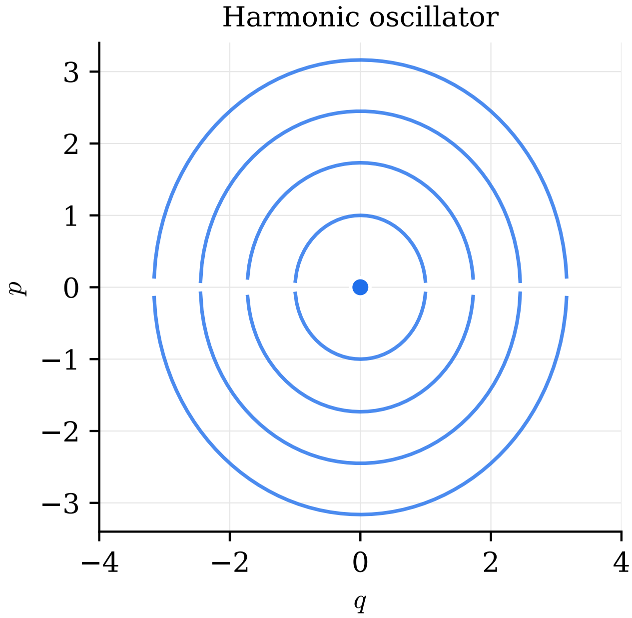

Figure 1: The equations of motion as a vector field; a trajectory is everywhere tangent to it (here, the nested ellipses of the harmonic oscillator).

2 The harmonic oscillator

Last lesson we solved the oscillator outright: with $U = \frac{1}{2}m\omega^2 q^2$, Newton's law $\ddot{q} = -\omega^2 q$ has the solution $q(t) = q(0)\cos\omega t + \dot{q}(0)\sin(\omega t)/\omega$. Read it in the phase plane. Releasing *from rest* at $q(0) = A$ (so $\dot{q}(0) = 0$ and the momentum $p(0) = 0$) gives $q(t) = A\cos\omega t$ and $p(t) = m\dot{q}(t) = -m\omega A\sin\omega t$, so the phase point $x(t) = (q, p)$ visits

t	0	$\pi/2\omega$	π/ω	$3\pi/2\omega$
$x = (q, p)$	$(A, 0)$	$(0, -m\omega A)$	$(-A, 0)$	$(0, +m\omega A)$

right, bottom, left, top: the state runs clockwise around an ellipse, closing at $t = 2\pi/\omega$. This is no accident, and it holds for *any* start. For general initial data (q_0, p_0) the solution is $q(t) = q_0\cos\omega t + \frac{p_0}{m\omega}\sin\omega t$, $p(t) = p_0\cos\omega t - m\omega q_0\sin\omega t$; substituting into the energy and using $\cos^2(\omega t) + \sin^2(\omega t) = 1$, the time cancels and

$$E\big(q(t), p(t)\big) = \frac{p(t)^2}{2m} + \tfrac{1}{2}m\omega^2 q(t)^2 = \frac{p_0^2}{2m} + \tfrac{1}{2}m\omega^2 q_0^2,$$

the same at every instant: the energy is conserved along every solution, for every (q_0, p_0). Each trajectory is therefore confined to a level set $E(q, p) = \text{const}$, the equation of an ellipse,

$$\frac{p^2}{2m} + \tfrac{1}{2}m\omega^2 q^2 = E \iff \frac{q^2}{2E/(m\omega^2)} + \frac{p^2}{2mE} = 1.$$

And here is the point we keep: *we did not really need the solution*. Conservation of energy alone forces every trajectory onto a level curve $E(q, p) = \text{const}$; their whole family, one ellipse per energy, nested about the origin and run clockwise, is the *phase portrait*, with semi-axes $\sqrt{2E/(m\omega^2)}$ along q and $\sqrt{2mE}$ along p. Where the ellipse meets the q-axis the particle is momentarily at rest (the *turning points*, $U(q) = E$); at the bottom of the well the energy is all kinetic; the centre $(0, 0)$ is the equilibrium. A phase portrait has discarded the clock, so it no longer says *when* the particle is where; but it shows *all* motions at once, one curve per energy.

3 The pendulum

The method comes into its own where solutions do not exist. The plane pendulum of length ℓ obeys the nonlinear equation

$$\ddot{\theta} + \omega_0^2 \sin\theta = 0, \qquad \omega_0^2 = g/\ell,$$

with no solution in elementary functions, yet its portrait is immediate. With $U(\theta) = \omega_0^2(1 - \cos\theta)$ (and $p = \dot{\theta}$, the momentum conjugate to θ in units where the moment of inertia $m\ell^2 = 1$) the conserved energy gives $p(\theta) = \pm\sqrt{2\,(E - \omega_0^2(1 - \cos\theta))}$, read straight off the corrugated potential (Figure 2). Three regimes appear: *small oscillations* near the bottom, where $\sin\theta \approx \theta$ and the pendulum is a harmonic oscillator; *libration* $(E < 2\omega_0^2)$, swinging between turning points on a closed orbit; the *separatrix* $(E = 2\omega_0^2)$, $p = \pm 2\omega_0 \cos(\theta/2)$, through the upright $\theta = \pi$; and *rotation* $(E > 2\omega_0^2)$, swinging over the top.

The saddle and the no-crossing rule. The upright $(\theta, p) = (\pi, 0)$ is a maximum of U, hence an equilibrium and its own trajectory; the separatrix only *approaches* it, taking infinite time, so its apparent self-crossing is not a true one.

The shape of phase space. Since θ and $\theta + 2\pi$ are the same configuration, the left and right edges are identified and the phase plane becomes a *cylinder*, $\Gamma = S^1 \times \mathbb{R}$ (Figure 3). On it a rotation is a loop that wraps around and closes, a libration a small loop that does not, and the two saddles merge into a single point on the seam.

Remark 1 (An analytic solution does exist). Although elementary functions fail, the pendulum can be solved exactly with *Jacobi elliptic functions*: for libration of amplitude $\theta_{\max}$, $\sin(\theta/2) = k\,\mathrm{sn}(\omega_0 t, k)$ with $k = \sin(\theta_{\max}/2)$, and the period is $T = 4K(k)/\omega_0$, with K the complete elliptic integral of the first kind. It reduces to $2\pi/\omega_0$ as $k \to 0$ and diverges as $k \to 1$ (the separatrix), exactly what the portrait shows at a glance, without the formula.

4 The phase flow: existence, uniqueness, no crossing

In both examples we quietly relied on something: distinct trajectories never meet, and the pendulum's separatrix only *approaches* the upright point without ever crossing. This is no accident; it is the content of one theorem, the foundation under every portrait we drew (proof in Appendix A).

Theorem 1 (Existence and uniqueness; Picard–Lindelöf)**.** *If* $\mathbf{V}$ *is continuously differentiable* (C^1) *and autonomous* ($\mathbf{V} = \mathbf{V}(x)$, *with no explicit time* t), *then through every point* $x_0 \in \Gamma$ *there passes exactly one trajectory* $x(t)$ *with* $x(0) = x_0$.

The structural fact that makes a phase portrait legible is that **orbits never cross**.

Remark 2 (Orbits never cross). Through every point of phase space passes exactly one orbit; distinct orbits never meet, and none crosses itself except by closing into a loop. A crossing would be a single state with two futures, which uniqueness forbids. This holds because the field is *autonomous*: $\mathbf{V}(x)$ depends on the state, not on time t, so each point carries a single velocity and hence a single future.[1]

[1] More precisely, the theorem gives a unique *solution* $x(t)$. Adjoining the time axis to phase space forms the *extended phase space* $\mathbb{R}_t \times \Gamma$, in which the solution sits as its graph $t \mapsto (t, x(t))$, an *integral curve*; distinct integral curves never meet. The *orbit* the portrait draws is this graph's projection onto Γ, and orbits avoid crossing only because the field is autonomous; for a time-dependent $\mathbf{V}(x,t)$ the projections may cross while the graphs stay disjoint, the clean picture being restored there by the autonomous extension $(\dot{x}, \dot{t}) = (\mathbf{V}, 1)$.

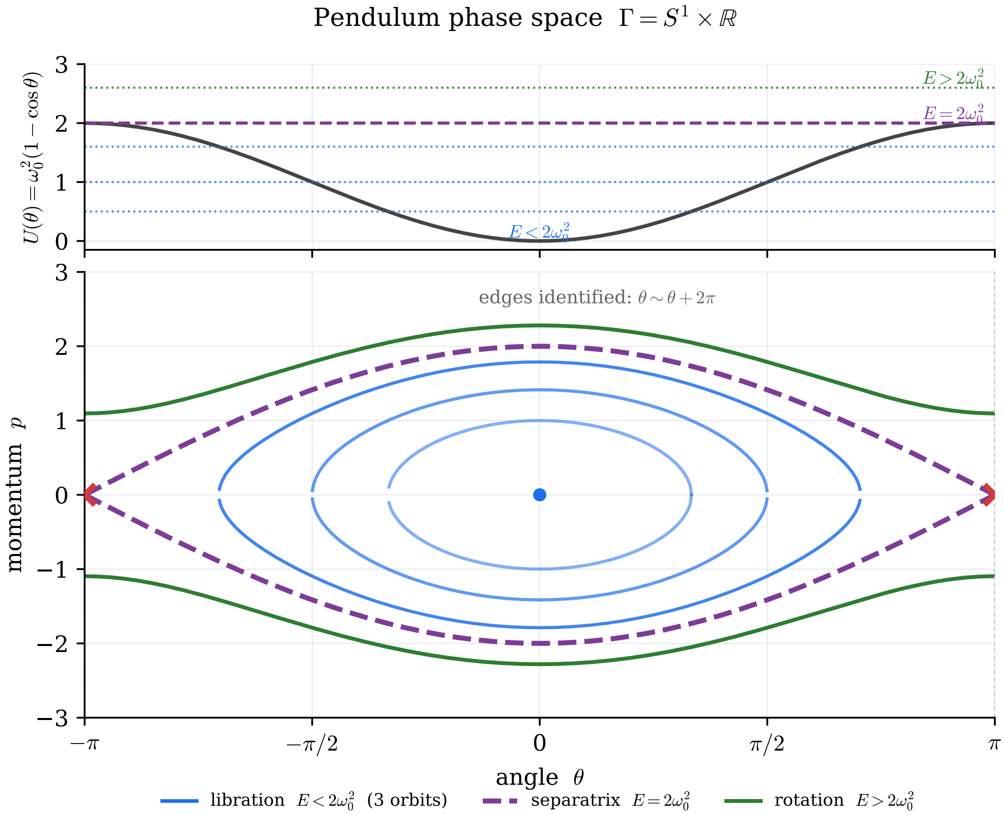

Figure 2: The pendulum. The potential $U(\theta)$ with three energy levels (top) and the portrait (bottom): centres at $\theta = 0, \pm 2\pi$, saddles at $\theta = \pm\pi$; inside the separatrix, libration; outside, rotation. Periodic with period 2π.

Two further remarks round this out. *Determinism:* the state fixes the entire future and, run backward, the past. *Equilibria:* a point with $\mathbf{V} = 0$ is a motionless trajectory, which needs $p = 0$ and $F(q) = 0$, i.e. q at a critical point of U.

Remark 3 (Determinism is not predictability). Uniqueness is *local*, a solution may escape to infinity in finite time ($\dot{x} = x^2$), and although the flow depends continuously on x_0 over finite times, nearby trajectories can separate exponentially over long ones: deterministic yet unpredictable. That gap is the seed of chaos. Determinism is also *contingent* on the smoothness of $\mathbf{V}$: if it is merely continuous, uniqueness can fail ($\dot{x} = \sqrt{|x|}$; "Norton's dome"), see Appendix A.

Outlook

Next lesson we will show that the same reading works for *any* one-dimensional conservative system, with no solving: every trajectory lies on a level curve of the energy,

$$\boxed{p(q) = \pm\sqrt{2m\,[\,E - U(q)\,]},}$$

read straight off the graph of U: the motion confined to $U(q) \le E$, every minimum of U a **centre** (expanded, a harmonic oscillator of frequency $\omega = \sqrt{U''(q^*)/m}$, so *every minimum hides an oscillator*), every maximum a **saddle** with its separatrix at $E = U(q_{\max})$. The quartic double well makes it concrete.

Example 1 (The quartic double well). For $U(q) = \frac{1}{4}q^4 - \frac{1}{2}q^2$, $U'(q) = q(q-1)(q+1)$, so the critical points are $q = 0, \pm 1$ with $U'' = 3q^2 - 1$. Thus $q = \pm 1$ are centres (minima) and $q = 0$ a saddle (maximum); the separatrix energy is $E = U(0) = 0$. For $-\frac{1}{4} < E < 0$ the allowed region

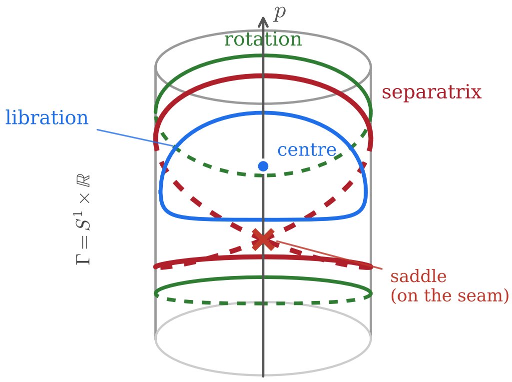

Figure 3: The same portrait on its cylinder $\Gamma = S^1 \times \mathbb{R}$. The separatrix (red) is two loops meeting at the single saddle on the back seam; libration (blue) sits inside on the front, rotation (green) wraps around outside.

is two disjoint wells (two closed orbits, the particle trapped in one); for $E > 0$ a single orbit encloses both; at $E = 0$ the separatrix is the figure-eight through the saddle (Figure 4). Unlike the pendulum, every orbit is bounded and Γ is the plane.

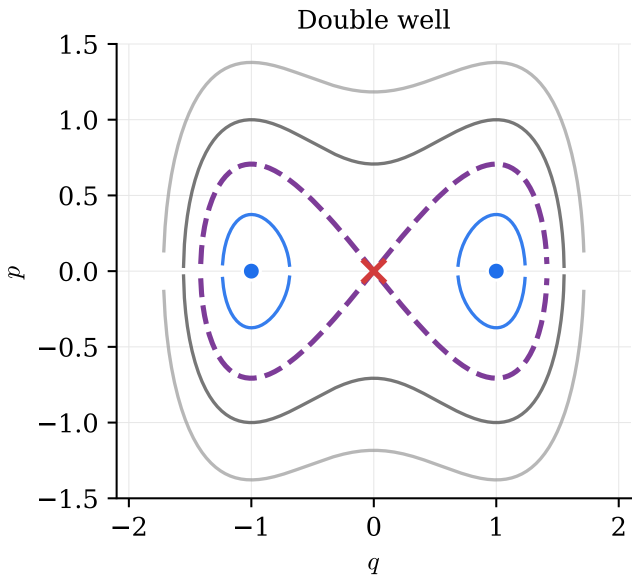

Figure 4: The quartic double well: two centres, one saddle, and the figure-eight separatrix (purple) dividing single-well oscillation from the large orbit over the barrier.

Turning that qualitative reading into numbers, the period $T(E) = \sqrt{2m}\int_{q_-}^{q_+} \mathrm{d}q/\sqrt{E - U(q)}$ of a closed orbit, the energy method for an arbitrary $U(q)$, and the way several wells and their separatrices partition the plane, is where we pick up.

Looking ahead

Step back: we asked only what the *state* of a system is, and we can now read every motion of a one-dimensional system off the geometry of a single picture, without solving anything. Phase space is not merely a convenient picture; it is the natural home of mechanics and of much beyond it. *Hamiltonian mechanics:* the first-order flow $(\dot{\mathbf{q}}, \dot{\mathbf{p}})$ is the seed of Hamilton's equations, and the enclosed area $\oint \mathbf{p} \cdot \mathrm{d}\mathbf{q}$ their central invariant, the systematic theory we build next. *Chaos:* sensitive dependence and mixing are statements about phase-space geometry. *Statistical mechanics:* a macroscopic system is one point in a phase space of enormous dimension, an ensemble a cloud carried by the flow, and entropy the volume it occupies. *Quantum mechanics:* the uncertainty principle forbids a sharp point, assigning each state a minimal area of order $\hbar$, the meeting point of the classical and quantum descriptions.

A Existence and uniqueness, by the contraction mapping theorem

We prove the existence–uniqueness theorem of §4 in any dimension, $x \in \mathbb{R}^n$ (one particle on a line is $n = 2$, $x = (q, p)$). The hypothesis is that $\mathbf{V}$ be *Lipschitz* near x_0, $\|\mathbf{V}(x) - \mathbf{V}(y)\| \leq L\|x - y\|$; this is automatic when $\mathbf{V}$ is C^1, a bounded derivative supplying the constant L. It cannot be relaxed to mere continuity: for $\dot{x} = \sqrt{|x|}$, $x(0) = 0$ the field is continuous but not Lipschitz at 0, and uniqueness fails (both $x \equiv 0$ and $x = t^2/4$ solve it); continuity alone still buys *existence* (Peano), so the Lipschitz bound is precisely what secures *uniqueness*, the determinism of §4, and physically the absence of a Norton's dome.[2] Under it the IVP $\dot{x} = \mathbf{V}(x),\ x(t_0) = x_0$ has a unique solution on an interval about t_0.

Proof of the theorem. Throughout, $x_0 \in \mathbb{R}^n$ is the fixed initial point. A continuous $x(t)$ solves the IVP iff it solves the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} \mathbf{V}(x(s))\, \mathrm{d}s =: (Tx)(t).$$

Integrating $\dot{x} = \mathbf{V}(x)$ from t_0 to t, the fundamental theorem of calculus gives the left side $x(t) - x_0$ and hence this equation; conversely, differentiating it returns $\dot{x} = \mathbf{V}(x)$, and setting $t = t_0$ makes the integral vanish, recovering $x(t_0) = x_0$. A solution is thus exactly a fixed point of the *Picard operator* T. On a closed ball of radius r about x_0 where $\|\mathbf{V}\| \leq \mu$ and $\|\mathbf{V}(x) - \mathbf{V}(y)\| \leq L\|x - y\|$, set $M = \{x \in C(I; \mathbb{R}^n) : \|x - x_0\| \leq r\}$, $I = [t_0 - h, t_0 + h]$, complete in the sup-norm; choosing $h \leq r/\mu$ makes $T : M \to M$. Here, and only here, the Lipschitz bound is used,

$$\|(Tx)(t) - (Ty)(t)\| \leq \int_{t_0}^{t} L\, \|x(s) - y(s)\|\, \mathrm{d}s \leq L\, h\, \|x - y\|,$$

so for $h < 1/L$, $\kappa := Lh < 1$ and T is a contraction. The contraction-mapping lemma below (Banach–Caccioppoli) then gives the unique fixed point, the unique local solution. □

Two refinements. The Picard iterates are explicit and converge on the *whole* interval, not merely a short one: from $x_0(t) \equiv x_0$, $\|x_{n+1}(t) - x_n(t)\| \leq \mu L^n |t - t_0|^{n+1}/(n+1)!$, and the *factorial* (the series that builds e^{Lt}) converges for every t. Uniqueness can also be seen directly, bypassing the fixed point: if x, y both solve the IVP, $\Delta(t) = \|x(t) - y(t)\|$ obeys $\Delta(t) \leq L \int_{t_0}^{t} \Delta(s)\, \mathrm{d}s$, and Grönwall's inequality forces $\Delta \equiv 0$.

It remains to establish the tool we invoked.

Theorem 2 (Banach–Caccioppoli, for Euclidean curves)**.** *Work on the continuous curves $u : I \to \mathbb{R}^n$ with the sup-norm $\|u\| = \max_{t \in I} \|u(t)\|$, so the distance between two curves is $\|u - v\|$. Since $\mathbb{R}^n$ is complete, so is this space of curves; a map T with $\|Tu - Tv\| \leq \kappa\, \|u - v\|$, $0 \leq \kappa < 1$, then has exactly one fixed point, the limit of the iterates $u_{n+1} = Tu_n$ from any start.*[3]

[2] When Γ is a manifold rather than $\mathbb{R}^n$, as for the pendulum's cylinder $S^1 \times \mathbb{R}$, the statement still holds: it is local, every chart turns a patch of Γ into an open piece of $\mathbb{R}^n$ and the argument runs there unchanged, while the identification $\theta \sim \theta + 2\pi$ governs only how trajectories close up, not their local existence or uniqueness.

[3] This lemma has a more general form, valid for contractions on any complete metric space; here we use only the Euclidean case.

Proof of the lemma. Fix any curve u_0 and iterate $u_{n+1} = Tu_n$. Each step contracts, $\|u_{n+1} - u_n\| = \|Tu_n - Tu_{n-1}\| \leq \kappa \|u_n - u_{n-1}\| \leq \kappa^n \|u_1 - u_0\|$, so for $m > n$ the triangle inequality and the geometric series give

$$\|u_m - u_n\| \leq \sum_{k=n}^{m-1} \|u_{k+1} - u_k\| \leq \|u_1 - u_0\| \sum_{k=n}^{m-1} \kappa^k \leq \frac{\kappa^n}{1-\kappa} \|u_1 - u_0\|,$$

which tends to 0 as $n \to \infty$. The iterates therefore form a **Cauchy sequence** (their terms get arbitrarily close to one another), and by **completeness** of the curve space they converge to some curve $u_\star$. A contraction is continuous, $\|Tu_n - Tu_\star\| \leq \kappa \|u_n - u_\star\| \to 0$, so letting $n \to \infty$ in $u_{n+1} = Tu_n$ gives $u_\star = Tu_\star$: a fixed point. It is the only one, since two fixed points would obey $\|u_\star - v_\star\| = \|Tu_\star - Tv_\star\| \leq \kappa \|u_\star - v_\star\|$, impossible for $\kappa < 1$ unless $u_\star = v_\star$. □